

## VECTOR POTENTIAL METHODS

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## SUMMARY

In this paper vector potential and related methods, for the simulation of both inviscid and viscous flows over aerodynamic configurations, are briefly reviewed. The advantages and the disadvantages of several formulations are discussed and alternate strategies are recommended. The paper consists of the following sections

Introduction

Scalar Potential

Modified Potential

Alternate Formulations of Euler Equations

Least-Squares Formulation

Variational Principles

Remarks on Iterative Techniques and Related Methods

Viscous Flow Simulation

Conclusions

## INTRODUCTION

Most of the recent successful attempts to solve Euler equations are based on the primitive variable formulation where the governing equations are written in a divergence form representing the conservation laws of mass, momentum and energy. The main reason is the capability of correctly capturing flow discontinuities (shocks and wakes) with the help of artificial viscosity terms, explicitly added to the equations or implicitly built in the numerical schemes. Moreover, the conservation laws are basically a hyperbolic system of equations which can be easily integrated in time (using for example explicit schemes like Runge Kutta methods) to obtain the steady state solution. This fact is particularly attractive if unstructured grids are used for simulating flows over complex three-dimensional configurations. Such a calculation can be very efficient on present computers since it is fully vectorizable and indeed very impressive results were obtained using multigrid convergence acceleration techniques.

Artificial viscosity may lead, however, to artificial vorticity, artificial boundary layers and separation. To minimize the contamination of the inviscid solution with such artificial viscous effects, higher order schemes are preferred and to avoid overshoots and undershoots near shocks, flux limiters are applied. Finite-difference schemes, with the total variation diminishing property, are proposed. Because of their diagonal dominance, relaxation methods are applicable and hence they are used as smoothers for multigrid techniques. The limiters, however, are highly nonlinear and some difficulties are expected, such as nonexistence and nonuniqueness of the discrete solution of the conservation laws, as well as slow

convergence and limit cycles. Research efforts will continue to construct high resolution schemes particularly to capture contact discontinuities, and successful applications of multigrid acceleration techniques will follow.

The purpose of the present paper is to examine alternate approaches, particularly for aerodynamic applications. The rationale, here, is very simple; far away from the body, the disturbances are small and the flow can be adequately described by a single equation, with a single unknown (the potential function) and a single parameter (Mach number); the potential formulation is valid, as long as the vorticity in the field is negligible.

One obvious choice is the zonal approach, where the potential formulation is restricted to the irrotational flow zone while the Euler primitive variable formulation is used for the inviscid rotational flow zone. One problem with this approach is the matching of the two local solutions along the interface or in the overlap domain of the two zones. The quality of the solution and the convergence of the overall calculations will depend on the numerical treatment of the interface problem.

Another approach is to augment the potential formulation with the rotational effects. This can be done using the Helmholtz decomposition theorem, where the velocity vector can be represented as the sum of a gradient of a scalar potential function and the curl of a divergence free vector. The correction to the potential formulation automatically vanishes, when the flow is irrotational. The problem with this formulation is the implementation of proper boundary conditions for the corrections particularly for multiply connected domains.

Related to this approach is the use of a variational principle and the Clebsch transformation. The connections to least-squares formulations are also delineated.

Extensions of these methods for viscous flow simulations are discussed and the relation to some viscous/inviscid interaction procedures are depicted.

In the following, recent work on scalar potential methods are reviewed first and then details of the above approaches are studied. Finally, some concluding remarks are drawn.

### SCALAR POTENTIAL

In the recent literature, one can find excellent reviews of methods of solution of the potential equation (See for example Caughey (1), Holst (2) and South (3)). In this section, only the most recent developments will be reviewed.

Steady inviscid flows, with constant entropy and total enthalpy, can be described by a potential function satisfying the continuity equation in conservation form

$$\nabla \cdot (\rho \nabla \phi) = 0 \quad (1)$$

where

$$\rho = \left(1 - \frac{\gamma-1}{2} M_\infty^2 (|\nabla \phi|^2 - 1)\right)^{\frac{1}{\gamma-1}} \quad (1')$$

and  $M_\infty$  is the free stream Mach number.

The full potential equation (1) is nonlinear and of mixed type (elliptic in the subsonic region and hyperbolic in the supersonic region). It admits expansion as well as compression shocks. Honest discretization of equation (1) leads to a discrete system of equations with the same properties. One way to exclude expansion shocks is to use artificial viscosity methods. No vorticity can be generated due to the artificial viscosity in such calculations since the velocity field is calculated as a gradient of the potential function. Moreover, for pure subsonic flows, no artificial viscosity is needed and the potential solution should satisfy the Euler equations. On the other hand, to obtain the solution of Euler equations, artificial viscosity is usually needed even for pure subsonic flows.

The artificial viscosity can be introduced in the numerical scheme in many ways. Osher et al. (4) introduced a flux biasing scheme which, unlike the retarded density scheme, does not allow for expansion shocks in the discrete solution of equation (1). This scheme is first-order accurate in the supersonic region, while the artificial viscosity is switched off in the subsonic region. Shankar et al. (5) used it successfully for time-dependent calculations. Volpe and Jameson (6) applied a second-order version of the same scheme in their multigrid code and obtained impressive results.

Recently, Mostrel (7) introduced a second-order accurate scheme, with no subsonic/supersonic switching, and proved global linear stability, total variation diminishing (with flux limiters) and discrete entropy inequality. He also introduced a time splitting algorithm for the 2D low-frequency transonic small disturbance equation. In this area, one should mention the unpublished work of Catherall\* who introduced an approximate factorization scheme, AF2, which consists of two factors only, even for three-dimensional flows.

Various methods are used to accelerate the convergence of potential flow calculations. Besides multigrid, generalized conjugate gradients, generalized minimal residuals (GMRES) and extrapolation procedures can be applied on computers with large memories. For two dimensional (and axisymmetric) problems, direct solvers based on banded Gaussian elimination or nested dissection are nowadays feasible and quadratic convergence can be obtained using Newton's method (8-9). For three dimensional problems, block relaxation and domain decomposition procedures are needed. Some of these algorithms are suitable for parallel processors.

Few years ago, many three-dimensional codes were developed for potential flow calculations. There are two common mistakes in most of these codes. The first is to use a two-dimensional vortex as a far-field boundary condition for each spanwise station. The second is to assume that the flow leaves the airfoil section in a direction bisecting the trailing-edge angle as in two-dimensional problems; this is not true for general three-dimensional lifting flows. In all these codes, the wakes are fixed and are not allowed to adjust with the flow. No effort has been spent to correct these codes and it seems there is no interest to do so! In Europe, a finite-element code based on optimal control formulation was developed at the same time to simulate flow over a complete aircraft. It is not clear, however, how the vortex sheets are treated in such calculations.

Recently, an intensive effort was launched at Boeing to develop a similar code (TRANAIR). Some of its innovative ideas are discussed in ref. (10). (An interesting test case is a wing in an incompressible flow where panel methods can

\*Catherall, D., private communication, July 1985.

predict correctly the induced drag. Moreover, theoretical results are available for certain configurations.)

Finally, no progress has been made to further understand the nonuniqueness problem (11) of the lifting potential flows in the transonic regime. Since no multiple solutions of the Euler equations for the lifting airfoil in the transonic regime have been reported, one tends to assume that the potential model is inadequate. It seems, however, that acceptable results can be obtained if boundary layer interactions are taken into account. After all, the potential flow approximation is only meaningful outside the viscous layers. It seems also that the viscous transonic equation (with uniform viscosity) has a unique solution but no proof is available.

### MODIFIED POTENTIAL

In general, due to the absence of entropy and vorticity effects, shocks in potential solutions are stronger. In this section, approximate methods to account for these effects are discussed.

It is argued that the vorticity effects are higher order. In fact, for straight shocks there is an entropy jump, but there is no entropy gradient downstream of the shock and, therefore, no vorticity is generated (it is well known that vorticity is related to the curvature of the shock). If the vorticity is neglected, the flow downstream of a shock can be also described by a potential function. The flow, however, is no longer isentropic. In this model, the entropy jump is accounted for by modifying the shock point operator. It turns out that the equation downstream of the shock is unaltered (since entropy is constant along streamlines). This approximation is completely equivalent to fit a Rankine-Hugoniot shock in potential flows. Special treatment of wakes may be also required (12), (13).

The next step is to construct a simple approximation for the vorticity effects. This can be achieved by augmenting the velocity due to the potential field by a rotational increment due to the entropy gradient. In two dimensions, an ordinary differential equation is solved for the rotational component,  $\bar{u}$ , namely

$$\frac{\partial \bar{u}}{\partial n} = - \rho \frac{d(\Delta S/R)}{d\psi} \quad (2)$$

For more details see ref. (12).

An approximate solution of equation (2) is simply

$$\bar{u} = - \frac{\rho}{\rho q} \frac{\Delta S}{R} = - \frac{\Delta S/R}{\gamma M_\infty^2} \quad (2')$$

Although the rotational correction is formally higher order than the correction due to the modification of the jump condition across the shock\*, it is not recommended to neglect  $\bar{u}$ , particularly for lifting airfoils.

The reason is clear from examining the work of Klopfer and Nixon (14). Across the wake, the static pressure must be continuous while the total pressure, in general, is not. Therefore, the tangential velocity component must jump. If the flow is presented only by a potential function, the jump of the potential across the wake will grow linearly with the distance from the trailing edge (i.e. in the far field,  $r \rightarrow \infty$ !).

On the other hand, this nonuniformity does not occur if the potential field is augmented by the rotational component. In this case, both the entropy and  $\bar{u}$  jump across the wake while  $\phi_x$  does not. This can be shown by expanding the static pressure formula (assuming the wake is aligned with the x - axis)

$$p = (1 - \frac{\gamma-1}{2} M_\infty^2 ((\phi_x + \bar{u})^2 - 1))^{\frac{\gamma}{\gamma-1}} \cdot e^{-\Delta S/R} / \gamma M_\infty^2 \quad (3)$$

The contribution of the entropy term cancels the contribution of the rotational component and therefore  $\phi_x$  must be continuous to guarantee continuous static pressure across the wake.

For the three-dimensional flows, the calculations of the rotational velocity components are more complicated as will be discussed in the next sections. As a crude approximation, the two-dimensional formula can be used in each spanwise station. Dang and Chen (15) used instead, the following formula

$$\vec{q} = \nabla \phi - \frac{\Delta S/R}{\gamma M_\infty^2 (n_x \cos \alpha + n_y \sin \alpha)} \hat{n} \quad (4)$$

where  $\hat{n} = n_x \hat{e}_x + n_y \hat{e}_y$  is normal to the shock surface. In our opinion, the extra computational effort in the calculation of  $\hat{n}$  is not justifiable and the assumption of a locally normal shock is consistent with the other approximations made in the derivation of the rotational velocity component formulas ((2') as well as (4)).

\* From Prandtl relation,  $(1+u_1) \cdot (1+u_2) = a^*{}^2$ , the correction to the shock jump condition to allow for entropy generation is second order, while  $\bar{u}$  is third order.

## ALTERNATE FORMULATIONS OF EULER EQUATIONS

### Two-Dimensional Flows

In this section, no small disturbance approximations are assumed and the treatment of the exact inviscid equations is discussed. It is instructive to consider first a simple case of two-dimensional inviscid incompressible steady flow. The conservation laws are usually written in terms of the continuity and the momentum equations. On the other hand, the following equations are completely equivalent:

$$\begin{aligned} u_x + v_y &= 0 \\ -u_y + v_x &= \omega(\psi) \end{aligned}$$

$$\frac{P}{\rho} + \frac{1}{2} (u^2 + v^2) = H(\psi) \quad (5)$$

where  $\omega$  and  $H$  are constants along streamlines. Equation (5) represents conservation of mass, vorticity and total enthalpy. The first two equations are linear in  $u$  and  $v$ ; the pressure is decoupled and is obtained from Bernoulli's law. If equation (5) is used, contact discontinuities (wakes) must be fitted and no artificial viscosity is needed in such calculations.

The exact equation for two-dimensional inviscid compressible flows is

$$W_t + F_x + G_y = 0 \quad (6)$$

where  $W = (\rho, \rho u, \rho v, \rho e)^T$  and  $F$  and  $G$  are the corresponding flux vectors. The Jacobian of  $F$  has the eigenvalues  $u \pm a, 0, 0$ , and similarly the Jacobian of  $G$  has the eigenvalues  $v \pm a, 0, 0$ , where  $a$  is the speed of sound. Four modes are identifiable: two acoustic modes, entropy mode and vorticity mode. In fact, for steady flows, the above equations can be rewritten in the following nonconservative form:

$$(a^2 - u^2) u_x - uvv_x - uvu_y + (a^2 - v^2) v_y = 0 \quad (6a)$$

$$-u_y + v_x = \omega \quad (6b)$$

$$\frac{D\Delta S/R}{Dt} = 0 \quad (6c)$$

$$\frac{DH}{Dt} = 0 \quad (6d)$$

where  $\frac{D}{Dt}$  is the substantial derivative.

The vorticity  $\omega$  is calculated from the Crocco's relation

$$\omega = \rho \frac{d\Delta S/R}{d\psi} - \rho \frac{dH}{d\psi} \quad (7)$$

The speed of sound is given by

$$a^2 = (\gamma - 1) \left( H - \frac{q^2}{2} \right) \quad (8)$$

where  $q^2 = u^2 + v^2$ , and the density and the pressure relations are

$$\rho = (M_\infty^2 a^2)^{\frac{1}{\gamma-1}} \cdot e^{-\Delta S/R} \quad (9)$$

$$p = (M_\infty^2 a^2)^{\frac{\gamma}{\gamma-1}} \cdot e^{-\Delta S/R} / \gamma M_\infty^2 \quad (9')$$

$\Delta S/R$  must have the proper jump across the shock to satisfy the Rankine Hugoniot relations.

It is clear from equations (6a) and (6b) that the corresponding characteristics have the same form for rotational and irrotational flows.

A shock fitting procedure must be used with this nonconservative formulation. In refs. (12), (16), the author used the Prandtl relations across the shock

$$q_{1n} q_{2n} = a^*{}^2 - \frac{\gamma-1}{\gamma+1} q_t^2 \quad (10a)$$

$$q_{t_1} = q_{t_2} = q_t \quad (10b)$$

For flows with constant entropy and total enthalpy, the vorticity vanishes and the irrotational motion can be represented by a potential field. The weak solution admitted by the potential equation (1) reflects conservation of mass under the isentropic assumption. On the other hand, in the modified potential formulation, the Prandtl relations are forced across the shock, which implies that the mass, including entropy effects, is conserved. To fully account for the entropy effects, a rotational velocity field due to the vorticity generated by the shock curvature must be added to the potential field; therefore, it is natural to use the decomposition

$$u = \phi_x + \tilde{u} \quad (11a)$$

$$v = \phi_y + \tilde{v} \quad (11b)$$

Obviously, such a decomposition is not unique; the rotational velocity components are not independent. A feasible constraint is

$$\tilde{u}_x + \tilde{v}_y = 0 \quad (12)$$

Such a constraint is automatically satisfied if a perturbative stream function  $\tilde{\psi}$  is introduced, such that

$$\tilde{u} = \tilde{\psi}_y \text{ and } \tilde{v} = -\tilde{\psi}_x \quad (13-14)$$

Thus, the governing equations for  $\phi$  and  $\tilde{\psi}$  are

$$(\rho\phi_x)_x + (\rho\phi_y)_y = -(\rho\tilde{\psi}_y)_x + (\rho\tilde{\psi}_x)_y \quad (15a)$$

$$\tilde{\psi}_{xx} + \tilde{\psi}_{yy} = -\omega \quad (15b)$$

Equations (12) & (13) are solved in ref. (13). Chaderjian and Steger (17) used a similar approach for the lifting airfoil problem, where both  $\phi$  and  $\tilde{\psi}$  are chosen to be continuous across the wake.

Another choice is

$$u = \phi_x + \tilde{\psi}_y/\rho \quad (16a)$$

$$v = \phi_y - \tilde{\psi}_x/\rho \quad (16b)$$

In this case the governing equations are

$$(\rho\phi_x)_x + (\rho\phi_y)_y = 0 \quad (17a)$$

$$(\tilde{\psi}_y/\rho)_y + (\tilde{\psi}_x/\rho)_y = -\omega \quad (17b)$$

With proper boundary conditions for  $\tilde{\psi}$ ,  $\phi$  can vanish identically. The solution of the transonic stream function equation is discussed in refs. (12) and (18). In ref. (19), Papailiou et al. used the decomposition

$$\rho u = \phi_x + \tilde{\psi}_y \quad (18a)$$

$$\rho v = \phi_y - \tilde{\psi}_x \quad (18b)$$

The corresponding equations are

$$\phi_{xx} + \phi_{yy} = 0 \quad (19a)$$

and 
$$(\tilde{\psi}_y/\rho)_y + (\tilde{\psi}_x/\rho)_x = -\omega \quad (19b)$$

In all these cases, a partial differential equation for the correction must be solved. Other decompositions require only the solution of ordinary differential equations. For example, the choice of  $\tilde{v} = 0$  in (11b) leads to

$$(\rho\phi_x)_x + (\rho\phi_y)_y = -(\rho\tilde{u})_x \quad (20a)$$

and 
$$\tilde{u}_y = -\omega \quad (20b)$$

For certain grids, aligned locally with the flow, this decomposition is obviously useful. Another example is based on a multiplicative correction,

$$u = \lambda\phi_x \quad (21a)$$

$$v = \lambda\phi_y \quad (21b)$$



The corresponding equations are

$$(\bar{\rho}\phi_x)_x + (\bar{\rho}\phi_y)_y = 0 \quad (22a)$$

$$\phi_x \lambda_y - \phi_y \lambda_x = -\omega \quad (22b)$$

where

$$\bar{\rho} = \lambda\rho \quad (23)$$

The general Clebsch representation has both additive and multiplication corrections, as will be discussed later. Based on this discussion, the full stream function formulation is the most recommended one. It is not widely used for transonic flows, however, because of the difficulties associated with the nonunique relationship between the density and the flux. The same remark holds for axisymmetric flows (including swirl).

### Three-Dimensional Flows

The conservation laws are given by

$$w_t + F_x + G_y + H_z = 0 \quad (5')$$

The eigenvalues of the Jacobians of F, G, and H are  $u \pm a, 0, 0, 0$ ,  $v \pm a, 0, 0, 0$  and  $w \pm a, 0, 0, 0$ . Acoustic, entropy and vorticity modes are still identifiable but the problem is more complicated since the vorticity has in general three nonzero components.

For steady flows, the governing equations can be rewritten in the form:

$$\nabla \cdot \rho \vec{q} = 0 \quad (24)$$

$$\nabla \times \vec{q} = \vec{\omega} \quad (25)$$

Assuming  $\rho$  and  $\vec{\omega}$  are given, equations (24) and (25) are four equations in three unknowns. There is, however, a relation between the vorticity components

$$\nabla \cdot \vec{\omega} = 0 \quad (26)$$

The Crocco relation is used to calculate two of the vorticity components

$$\vec{q} \times \vec{\omega} = -T\nabla S + \nabla H \quad (27)$$

The streamwise vorticity component can be calculated using equation (26). Alternatively, taking the cross product of (27) with  $\vec{q}$ , one can obtain the following formula for the vorticity (see refs. (21), (22), (24))

$$\vec{\omega} = \lambda \rho \vec{q} + \frac{\vec{q}}{|\vec{q}|^2} \times (T\nabla S - \nabla H) \quad (28)$$

Equation (26) leads to an equation for  $\lambda$

$$\rho \vec{q} \cdot \nabla \lambda = - \nabla \cdot \left[ \frac{\vec{q}}{|\vec{q}|^2} \times (\nabla S - \nabla H) \right] \quad (29)$$

The entropy and the total enthalpy transport equations are

$$\vec{q} \cdot \nabla S = 0 \quad (30)$$

$$\vec{q} \cdot \nabla H = 0 \quad (31)$$

While  $H$  and  $\lambda$  are continuous across a shock,  $S$  jumps. The speed of sound, the density and the pressure are obtained from equations (8), (9) and (9'), where  $q^2 = |\vec{q}|^2$ .

Using the Helmholtz theorem,  $\vec{q}$  can be decomposed into the gradient of a scalar function plus the curl of another vector

$$\vec{q} = \nabla \phi + \nabla \times \vec{\psi} \quad (32)$$

The first term on the right hand side of equation (27) is curl free, while the second term is divergence free. A feasible constraint on  $\vec{\psi}$  is

$$\nabla \cdot \vec{\psi} = 0 \quad (33)$$

Hence, equations (24) and (25) reduce to

$$\nabla \cdot (\rho \nabla \phi) = - \nabla \cdot (\rho \nabla \times \vec{\psi}) \quad (34)$$

$$\nabla^2 \vec{\psi} = - \vec{\omega} \quad (35)$$

The correction to the potential solution becomes a major effort, it requires the solution of three Poisson's equations in three dimensions.

The boundary condition for the potential problem is

$$\vec{n} \cdot \nabla \phi = - \vec{n} \cdot \nabla \times \vec{\psi} \quad (36)$$

where  $\vec{n}$  is normal to the solid surface. Two linear combinations of the stream functions can be kept constant on the boundary surface, and equations (33) can be used to solve for a third linear combination.

For example, in orthogonal coordinates, two components of  $\vec{\psi}$  can be chosen to be constant in the plane tangent to the body, hence  $\vec{n} \cdot \nabla \times \vec{\psi} = 0$ , while the normal derivative of the third component vanishes. The boundary conditions in generalized curvilinear coordinates are given in refs. (20), (21), for simply connected domains.

In a series of interesting papers (22), (23), (24), Dabaghi and Pironneau studied the vector potential method and obtained two- and three-dimensional finite-element solutions for transonic flow problems. The following variational problems are considered. For a given vector field  $\vec{q}$ , find  $\phi$  and  $\vec{\psi}$  such that:

$$(\nabla\phi, \nabla w) = (\vec{q}, \nabla w) \quad (37)$$

$$\begin{aligned} (\nabla x \vec{\psi}, \nabla x \vec{v}) + (\nabla \cdot \vec{\psi}, \nabla \cdot \vec{v}) \\ = (\vec{q}, \nabla x \vec{v}) \end{aligned} \quad (38)$$

where  $w$  and  $\vec{v}$  are proper weighting functions ( $\vec{v} x \vec{n}|_{\Gamma} = 0$ ,  $\int_{\Gamma} \vec{v} \cdot \vec{n} d\gamma = 0$ ;  $\Gamma$  is the boundary of a simply connected domain). It can be shown rigorously that the Helmholtz decomposition is unique

$$\vec{q} = \nabla\phi + \nabla x \vec{\psi} \text{ and } \nabla \cdot \vec{\psi} = 0 \quad (39)$$

The variational form of the vector identity

$$\nabla x \nabla x \vec{\psi} = -\Delta \vec{\psi} + \nabla \nabla \cdot \vec{\psi} \quad (40)$$

is given by

$$(\nabla \vec{\psi}, \nabla \vec{v}) = (\nabla x \vec{\psi}, \nabla x \vec{v}) + (\nabla \cdot \vec{\psi}, \nabla \cdot \vec{v}) + 2 \int_{\Gamma} \frac{\vec{\psi} \cdot \vec{v}}{R} d\gamma \quad (41)$$

where  $R$  is the mean radius of curvature of  $\Gamma$ . Therefore, equation (38) for  $\vec{\psi}$  is equivalent to

$$\begin{aligned} -\Delta \vec{\psi} &= \vec{\omega} \\ \frac{\partial \vec{\psi}}{\partial n} \cdot \vec{n} - 2 \frac{\vec{\psi} \cdot \vec{n}}{R} &= 0 \text{ on } \Gamma \\ \vec{\psi} \times \vec{n}|_{\Gamma} &= 0 \end{aligned} \quad (42)$$

In general, the three components of  $\vec{\psi}$  are coupled through the boundary conditions. The formulation has been extended to multiply connected domains by Dominguez (25). Across the wakes, the potential function jumps and the circulations (the difference in potential values) remain constant, while the normal component of the gradient of the potential is continuous. If the wake surface is denoted by  $\Sigma_j$  and the circulation by  $\lambda_j$  then

$$\left[ \phi \right]_{\Sigma_j} = \lambda_j, \quad \left[ \frac{\partial \phi}{\partial n} \right]_{\Sigma_j} = 0 \quad (43)$$

The following constraints must be imposed on  $\vec{\psi}$ :

$$\int_{\Gamma_i} \vec{\psi} \cdot \vec{n} d\gamma = \mu_i \quad (44)$$

where  $\mu_i$  are constants. The decomposition is no longer unique. It is unique, however, when  $\mu_i$  are given and  $\lambda_j$  are adjusted so that

$$\nabla\phi + \nabla x \vec{\psi} = \vec{q} \text{ on } \Sigma \quad (45)$$

The extra difficulty for nonsimply connected domains stems from the fact that

$$\nabla \times \vec{q} = 0, \quad \nabla \cdot \vec{q} = 0, \quad \vec{q} \cdot \vec{n}|_{\Gamma} = 0 \quad (46a)$$

does not imply  $\vec{q} = 0$  (see ref. (26)).

Unlike the primitive variable formulation of Euler equations, a special treatment of the Kutta condition (pressure is continuous at trailing edges) is required.

Another choice for the velocity decomposition is

$$\vec{q} = \nabla \phi + (\nabla \times \vec{\Psi})/\rho \quad (46b)$$

With proper nonhomogeneous boundary conditions for  $\vec{\Psi}$ ,  $\phi$  can vanish identically. This case is simply an extension of the stream function formulation to three dimensional flows. The equation for  $\vec{\Psi}$  is

$$\nabla \times (\nabla \times \vec{\Psi})/\rho = \vec{\omega} \quad (47)$$

and the boundary condition is

$$\vec{n} \cdot \nabla \times \vec{\Psi}|_{\Gamma} = \vec{q} \cdot \vec{n} \quad (48)$$

Hirasaki and Hellums (27)-(29) and Richardson and Cornish (29) studied a similar problem. A partial differential equation on the surface of the boundary must be solved. The corresponding variational formulation is given by Dabaghi and Pironneau (24). The problem is to find a function  $g$  such that

$$\vec{\Psi} \times \vec{n}|_{\Gamma} = g \quad (49)$$

implies equation (48).

It is shown that  $g = \nabla f$  where  $f|_{\Gamma}$  is the unique solution of a Laplace-Beltrami equation

$$\int_{\Gamma} \left( \frac{\partial f}{\partial s_1} \frac{\partial w}{\partial s_1} + \frac{\partial f}{\partial s_2} \frac{\partial w}{\partial s_2} \right) d\gamma = - \int_{\Gamma} \vec{q} \cdot \vec{n} w d\gamma \quad (50)$$

where  $w$  is a weighting function and  $(s_1, s_2)$  is a set of an orthogonal local coordinate system on  $\Gamma$ .

In ref. (20), Papailiou et al. used the decomposition

$$\rho \vec{q} = \nabla \phi + \nabla \times \vec{\Psi} \quad (51)$$

and  $\nabla \cdot \vec{\Psi} = 0$ .

Therefore, the equations for  $\phi$  and  $\vec{\Psi}$  are

$$\nabla^2 \phi = 0 \quad (52a)$$

$$-\nabla^2 \vec{\Psi} = \frac{1}{\rho} \nabla \rho \times (\rho \vec{q}) + \rho \vec{\omega} \quad (52b)$$

Lagging the density in the right hand side of equation (52b) may lead to convergence difficulties for transonic flow calculations.

In all these variations, the stream vector  $\vec{\psi}$  has three components. In ref. (30), another method is proposed. One of the components of  $\vec{\psi}$  is chosen to vanish identically, and viscous flows over a three-dimensional trough are calculated. The application to inviscid transonic flow calculations is given in ref. (31).

Giese (32) proposed also a representation of the flux vector in three dimensions using two scalar stream functions ( $\psi$  and  $\theta$ )

$$\rho \vec{q} = \nabla \psi \times \nabla \theta \quad (53)$$

The continuity equation is automatically satisfied since  $\nabla \cdot \rho \vec{q} = \nabla \cdot \nabla \psi \times \nabla \theta = 0$ . In this formulation, the body is a stream surface. The equations for  $\psi$  and  $\theta$  are obtained from

$$\nabla \times (\nabla \psi \times \nabla \theta / \rho) = \vec{\omega} \quad (54)$$

No numerical solutions are reported in the literature for transonic flow problems. In ref. (33) an application to hypersonic flows around a body of revolution at an angle of attack is discussed.

Recently, Rose (34) proposed to solve the Cauchy-Rieman equations using a finite element scheme where a three-dimensional potential solution in the element and only two dimensional stream function solutions on the boundary faces of the element, are required.

#### LEAST-SQUARES FORMULATIONS

In ref. (35), Fasel proposed to solve the incompressible flow equations using a velocity/vorticity formulation, where the pressure is eliminated. The equations  $\nabla \cdot \vec{q} = 0$  and  $\nabla \times \vec{q} = \vec{\omega}$  are replaced by

$$\nabla^2 \vec{q} = - \nabla \times \vec{\omega} \quad (55)$$

For viscous flow calculations,  $\vec{q} = 0$  on the solid surface and three Poisson's equations for three velocity components are solved, while  $\vec{\omega}$  is obtained from the vorticity transport equation. (Recently, Rose et al. (36) and Osswald et al. (37) solved the system of the first order equations directly).

It seems that, unlike the stream function formulations, there is no difficulty with the boundary conditions. This is not true in general, since conservation of mass is not explicitly imposed and careful treatment of boundary conditions is required, particularly for inviscid flows.

In ref. (38), the author proposed a least squares formulation with a systematic treatment of the boundary conditions. For the continuous problem, the following functional is minimized

$$I(\vec{q}) = \int_{\Omega} \alpha (\nabla \cdot \vec{q})^2 + |\nabla \times \vec{q} - \vec{\omega}|^2 d\Omega \quad (56)$$

where  $\alpha$  is a Lagrange multiplier. The functional  $I$  is discretized first on structured (or unstructured) grids. Minimization of the discrete version of  $I$ ,

with respect to the unknowns at each node leads to a conditioned system of algebraic equations.

To demonstrate the relation between the least-squares and the vector potential formulations, we consider first the two-dimensional incompressible flow case. The equations, in general, are

$$u_x + v_y = s \quad (57)$$

$$-u_y + v_x = \omega \quad (58)$$

where the  $s$  term in equations (57) represents sources or sinks in the field. Equations (57) and (58) are written in the form

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} s \\ \omega \end{pmatrix} \quad (59)$$

The least-squares formulation leads to the second order equations

$$L^* L \begin{pmatrix} u \\ v \end{pmatrix} = L^* \begin{pmatrix} s \\ \omega \end{pmatrix} \quad (60)$$

where  $L^*$  is the adjoint operator.

Alternatively, new variable  $\phi$  and  $\psi$  can be defined by the equation

$$L^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \quad (61)$$

Hence, equation (59) becomes

$$LL^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} s \\ \omega \end{pmatrix} \quad (62)$$

The variables  $\phi$  and  $\psi$  represents a potential and a stream function. Notice the right-hand side in equation (60) is differentiated while in equation (62), it is unaltered.

The extension to compressible three dimensional flows is given in ref. (39). In cartesian coordinates, the equations are

$$\begin{aligned} (\rho u)_x + (\rho v)_y + (\rho w)_z &= s \\ v_x - u_y &= \omega_3 \\ -w_x + u_z &= \omega_2 \\ w_y - v_z &= \omega_1 \end{aligned} \quad (63)$$

Equation (63) is written in the form

$$\begin{bmatrix} -\rho \partial_x & \partial_y \rho & \partial_z \rho \\ -\partial_y & \partial_x & 0 \\ \partial_z & 0 & -\partial_x \\ 0 & -\partial_z & \partial_y \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} s \\ \omega_3 \\ \omega_2 \\ \omega_1 \end{bmatrix} \quad (64)$$

The variables  $\phi$ ,  $\psi_1$ ,  $\psi_2$  and  $\psi_3$  are introduced via the adjoint operator.

$$\begin{bmatrix} -\rho \partial_x & \partial_y & -\partial_z & 0 \\ -\rho \partial_y & -\partial_x & 0 & \partial_z \\ -\rho \partial_z & 0 & \partial_x & -\partial_y \end{bmatrix} \begin{bmatrix} \phi \\ \psi_3 \\ \psi_2 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho v \\ \rho \omega \end{bmatrix} \quad (65)$$

or, 
$$-\rho \vec{q} = \rho \nabla \phi + \nabla \times \vec{\psi} \quad (66)$$

If  $s = 0$ ,  $\phi$  can identically vanish. Furthermore, if one of the  $\vec{\psi}$  components is chosen to be zero, a two-component stream function formulation can be obtained. Therefore, vector potential formulations are special cases of least squares.

## VARIATIONAL PRINCIPLES

### Unsteady Flows

When Lagrangian coordinates are used, the equations of motion can be derived from Hamilton's principle (that the difference of kinetic energy and potential energy be stationary). In the Eulerian description, Seliger and Whitham (40) considered the functional

$$\begin{aligned} \iint_R \left[ \left( \frac{1}{2} \rho q_i^2 - \rho e \right) + \phi \left( \frac{\partial \rho}{\partial t} + \frac{\partial (\rho q_i)}{\partial x_i} \right) \right. \\ \left. + \eta \left( \frac{\partial (\rho S)}{\partial t} + \frac{\partial (\rho q_i S)}{\partial x_i} \right) \right. \\ \left. + \beta \left( \frac{\partial (\rho \alpha)}{\partial t} + \frac{\partial (q_i \alpha)}{\partial x_i} \right) \right] dx dt \end{aligned} \quad (67)$$

where  $\phi$ ,  $\eta$  and  $\beta$  are Lagrange multipliers.

The variations with respect to  $q_i$ ,  $\rho$ ,  $S$ ,  $\alpha$  lead to

$$\begin{aligned} \delta q_i: q_i &= \frac{\partial \phi}{\partial x_i} + S \frac{\partial \eta}{\partial x_i} + \alpha \frac{\partial \beta}{\partial x_i} \\ \delta \rho: h &= \frac{1}{2} q_i^2 - \frac{D\phi}{Dt} - \alpha \frac{D\beta}{Dt} - S \frac{D\eta}{Dt} \\ \delta S: \frac{D\eta}{Dt} &= -T \end{aligned}$$

$$\delta\alpha: \frac{D\beta}{Dt} = 0 \quad (68)$$

where 
$$h = e + \frac{p}{\rho} = \frac{\gamma}{\gamma-1} \frac{p}{\rho}.$$

The variations of  $\phi, \eta$  and  $\beta$  reproduce the constraints

$$\delta\phi: \frac{\partial \rho}{\partial t} + \frac{\partial(\rho q_i)}{\partial x_i} = 0$$

$$\delta\eta: \frac{DS}{Dt} = 0$$

$$\delta\beta: \frac{D\alpha}{Dt} = 0 \quad (68')$$

The coordinate  $\alpha$  does not change along a particle path and the terms  $\alpha \nabla \beta$  in the velocity allow the representation of an initial vorticity to be separated from those produced by subsequent entropy gradients. In this representation the vorticity is given by

$$\vec{\omega} = \nabla S \times \nabla \eta + \nabla \alpha \times \nabla \beta \quad (69)$$

Seliger and Whitham (40) derived also a simplified form of the variational principle (67). Upon integration by parts the integrand  $L$  in (67) is equivalent to

$$L = \frac{1}{2} \rho q^2 - \rho e - \rho \frac{D\phi}{Dt} - \rho S \frac{D\eta}{Dt} - \rho \alpha \frac{D\beta}{Dt} \quad (70)$$

(all variations are taken to vanish on the boundary  $R$ )

From Equation (68),

$$L = \rho h - \rho e = p \quad (71)$$

Therefore, one can start directly with the functional

$$\iint_R p \, dX \, dt \quad (72)$$

where

$$p = p(h, S)$$

$$p_h = \rho, \quad p_S = -\rho T$$

Introducing a Clebsch representation for the velocity

$$\vec{q} = \nabla \phi + S \nabla \eta + \alpha \nabla \beta \quad (73)$$



where

$$h = -\phi_t - S\eta_t - \alpha\beta_t - \frac{1}{2}q_i^2 \quad (74)$$

the variations of the integral of the pressure with respect to  $\phi$ ,  $S$ ,  $\eta$ ,  $\alpha$  give exactly the corresponding equations of (68) and (68').

While the form in (72) is far from Hamilton's, it can be considered as a generalization of the well known Bateman principle of isentropic flows. It is noticed, however, that  $\alpha$ ,  $\beta$  are not uniquely defined by (73) since any perfect differential may be added to  $\phi$  with consequent changes in  $\alpha$  and  $\beta$ . From the equations of motion, it can be shown that

$$\frac{\partial (\Pi, \alpha, \beta)}{\partial (x, y, z)} = 0$$

where 
$$\Pi = h + \frac{1}{2}q_i^2 + \phi_t + S\eta_t + \alpha\beta_t \quad (75)$$

Therefore,  $\Pi = \Pi(\alpha, \beta, t)$  and  $\alpha$  and  $\beta$  can be obtained from the Hamiltonian form

$$\frac{D\alpha}{Dt} = -\frac{\partial \Pi}{\partial \beta}, \quad \frac{D\beta}{Dt} = \frac{\partial \Pi}{\partial \alpha} \quad (76)$$

It is sometimes advantageous to retain  $\Pi \neq 0$  and to use (75) in place of (74). The variational principle (72) is not modified but the variations with respect to  $\alpha$  and  $\beta$  then give (76) as required.

In fact, some of these ideas are very old. The velocity representation in (68) was first introduced by Clebsch in 1859, for the case of incompressible flow. Lamb (1932) considered a barotropic flow ( $p = p(\rho)$ ), where any velocity can be represented by

$$\vec{q} = \nabla\phi + \alpha\nabla\beta$$

and the momentum equations can be integrated to yield

$$\int \frac{dp}{\rho} + \frac{1}{2}q_i^2 = -\phi_t - \alpha\beta_t$$

where 
$$\frac{D\alpha}{Dt} = 0 \quad \text{and} \quad \frac{D\beta}{Dt} = 0.$$

The alternative Hamiltonian form of  $\alpha$  and  $\beta$  equations was first introduced by Stuart in 1900.

Recently, Buneman (41) showed that the flow equations can be put in a Hamiltonian form, with benefit for numerical schemes. He considered the isentropic flow case with Clebsch variables

$$\rho \vec{q} = -\sigma \nabla \mu - \rho \nabla \phi \quad (77)$$

Thus, the vortical and irrotational parts of the flow are represented in a symmetric manner. Note that there would be a similar third term if entropy changes are taken into account. In terms of  $\rho$ ,  $\phi$ ,  $\sigma$  and  $\mu$ , the following functional derivatives are obtained

$$\begin{aligned} \frac{\partial \Pi}{\partial \phi} &= \nabla \cdot \rho \vec{q} = -\frac{\partial \rho}{\partial t} \\ \frac{\partial \Pi}{\partial \rho} &= c \gamma \rho^{\gamma-1} - \frac{1}{2} \left(\frac{\sigma}{\rho}\right)^2 (\nabla \mu)^2 + \frac{1}{2} (\nabla \phi)^2 = \frac{\partial \phi}{\partial t} \\ \frac{\partial \Pi}{\partial \sigma} &= -\vec{q} \cdot \nabla \mu = \frac{\partial \mu}{\partial t} \\ \frac{\partial \Pi}{\partial \mu} &= -\nabla \cdot \sigma \vec{q} = -\frac{\partial \sigma}{\partial t} \end{aligned}$$

$$\text{where} \quad \Pi = C \rho^\gamma + (\sigma \nabla \mu + \rho \nabla \phi)^2 / 2\rho \quad (78)$$

and  $c$  is a constant.

The first equation is conservation of mass, and the second is a generalized Bernoulli equation. The third and the fourth equations are statements that the vortex parameters  $\mu$  and  $\sigma/\rho$  follow the flow. With this Hamiltonian form, one can stagger data for densities and potentials in space as well as in time. Also, densities can be updated by leap-frogging over potentials and vice-versa. No numerical solutions based on this formulation have been reported.

On the other hand, Ecer and Akay (42) were successful to calculate rotational and transonic flows using a variational principle similar to (67). They considered the functional

$$\begin{aligned} & \iint_R \left[ \left( \frac{1}{2} \text{eq}_i^2 - \rho e \right) + \phi \left( \frac{\partial \rho}{\partial t} + \frac{\partial (\rho q_i)}{\partial x_i} \right) \right. \\ & \quad \left. + \beta_k \rho \frac{D \alpha_k}{Dt} \right] dx dt \\ & + \iint_{\Gamma_1} f_\phi dr dt + \iint_{\Gamma_2} f_{\beta_k} dr dt \end{aligned} \quad (79)$$

where  $\Gamma_1$  and  $\Gamma_2$  are those portions of the boundaries where the following flux type quantities are specified.

$$f_{\phi} = \rho \vec{q} \cdot \vec{n} \quad (80)$$

$$f_{\beta_k} = \rho \alpha_k \vec{q} \cdot \vec{n} \quad (81)$$

The variations of (79) with respect to  $\alpha$  and  $\beta$  give

$$\delta\beta_k: \rho \frac{D\alpha_k}{Dt} = 0$$

$$\delta\alpha_k: \rho \frac{D\beta_k}{Dt} = \rho \frac{dH}{d\alpha_k} - \frac{p}{R} \frac{\partial S}{\partial \alpha_k} \quad (82)$$

Another choice is also suggested in ref. (42). The total enthalpy is employed as a primary variable to replace the material coordinate  $\alpha$  in (67). The proposed variational principle is, however, incorrect, since the energy equation for unsteady flow is

$$\frac{DH}{Dt} = \frac{1}{\rho} \frac{\partial p}{\partial t} \quad (83)$$

and  $H$  does change along a particle path!

### Steady Flows

Roberts (43) used directly the following decomposition for steady flows

$$\vec{q} = \vec{q}_0 + \nabla\phi + (S-S_0) \nabla\eta + (H-H_0) \nabla\beta \quad (84)$$

where the equations for  $\eta$  and  $\beta$  are given by

$$\frac{D\eta}{Dt} = -1 \quad (85)$$

$$\frac{D\beta}{Dt} = 1$$

For multiple shocks, multiple  $\eta$  fields must be introduced and the contribution to the velocity is given by

$$\sum_{t=1}^k (S_t - S_{t-1}) \nabla\eta_t \quad (86)$$

Similarly, multiple  $\beta$  fields must be introduced for multiple wakes. On the other hand, for isoenergetic flows,  $\beta$  vanishes identically. Grossman (44) calculated

successfully supersonic conical flows, with strong rotationality due to shocks, using such a decomposition.

For a thin body in a uniform flow, using small disturbance approximations, equation (85) becomes

$$u \frac{\partial \eta}{\partial x} = -T \quad (85')$$

substituting equation (85') into (84) yields

$$\begin{aligned} \vec{q} &\approx 1 + \nabla \phi \cdot \Delta S/R \frac{p}{\rho u} \vec{i} \\ &\approx 1 + \nabla \phi \cdot \frac{\Delta S/R}{\gamma M_\infty^2} \vec{i} \end{aligned} \quad (84')$$

The rotational component in equation (84') is identical with equation (2') which is derived based on Helmholtz decomposition.

A modified form of Clebsch representation was introduced by Hirsch et al. (45).

$$\vec{q} = \nabla \phi + \psi_1 \nabla S + \psi_2 \nabla H \quad (87)$$

Substituting equation (87) into the Crocco relation, one obtains for arbitrary and independent entropy and rothalpy gradients, the two equations for  $\psi_1$  and  $\psi_2$

$$\frac{D\psi_1}{Dt} = T$$

$$\frac{D\psi_2}{Dt} = -1 \quad (88)$$

A simplified representation can be obtained if a unique relation between S and H exists in the inlet flow field. In ref. (46), the authors discussed applications to rotational internal subsonic flows in ducts.

It is clear from the above discussion that the computational effort required for the variational principle formulation is less than that needed for the implementation of least squares or vector potentials. Thanks to Clebsch, we have a generalized Bernoulli equation for rotational, nonisentropic steady and unsteady flows.

#### REMARKS ON ITERATIVE TECHNIQUES AND RELATED METHODS

A common problem in all the various formulations discussed so far is the solution of a nonhomogeneous potential equation, where the right-hand side represents the rotational effects. A crucial point in developing an iterative technique to solve this nonlinear problem is to account for the mixed type nature of this equation.

Moreover, the characteristics of this equation should be based on the total velocity and not on the irrotational part only.

Techniques based on lagging the density as well as the rotational components are not reliable for transonic flow calculations. For example, the potential equation for two dimensional flows may be written in the form

$$(\rho^{n-1} \phi_x)_x + (\rho^{n-1} \phi_y)_y = -(\rho^{n-1} \bar{u})_x - (\rho^{n-1} \bar{v})_y \quad (89)$$

where  $n$  is the iteration count. Obviously, one should expect convergence difficulties since an asymmetric operator is replaced by a symmetric one. In practice, a crude solution may be obtained, if excessive artificial viscosity is used. To construct a scheme, independent of the artificial viscosity, equation (89) may be replaced by

$$\frac{\rho}{a^2} ((a^2 - u^2) \delta\phi_{xx} - 2uv \delta\phi_{xy} + (a^2 - v^2) \delta\phi_{yy}) = -(\rho u)_x - (\rho v)_y \quad (90)$$

It is not important to keep the left-hand side of equation (90) in conservative form and there are well-known methods to solve such a mixed type equation. Note, the potential correction will not affect the correction to the vorticity equation and ideally, the rotational components can be constructed such that they conserve mass. In fact, the Helmholtz decomposition can be applied to the correction rather than the original variables. To demonstrate this application of the Helmholtz theorem, consider the simple Cauchy-Riemann problem

$$\begin{aligned} u_x + v_y &= s \\ u_y - v_x &= -\omega \end{aligned} \quad (91)$$

One can construct the following two-step iterative technique

$$\begin{aligned} \text{step 1:} \quad \delta u_x + \delta v_y &= -(u_x + v_y - s) \\ \delta u_y - \delta v_x &= 0 \end{aligned} \quad (92)$$

$$\begin{aligned} \text{step 2:} \quad \delta u_x + \delta v_y &= 0 \\ \delta u_y - \delta v_x &= -(u_y - v_x + \omega) \end{aligned} \quad (93)$$

The correction in the first step is curl free, and according to the Stokes theorem, it can be represented by a potential. On the other hand, the correction in the second step is divergence free and according to the Gauss theorem, it can be represented by a stream function. (For a general three-dimensional problem, the correction in the second step can be represented as a curl of a divergence free vector.)

One can use Helmholtz decomposition directly on the discrete velocity field. A similar idea is behind the distributive relaxation discussed by Brandt in ref. (47). In such a technique, each discrete equation is satisfied in its turn by distributing changes to the several unknowns appearing in the equation in a specific manner; the main property is that in relaxing one equation, all the

residuals of the other equations are kept unchanged. Application to the Cauchy-Riemann problem on a staggered grid is discussed in ref. (47) and it is completely equivalent to a discrete Helmholtz decomposition.

Brandt also considered Stokes problem, where the above strategy is slightly modified. The equations are

$$\nabla \cdot \vec{q} = s \quad (94)$$

$$-\nabla \vec{q} + \nabla p = f \quad (95)$$

First, lagging the pressure, the velocity components can be updated from the momentum equations. In the second step, the velocity must be corrected to conserve mass (equation (94)). This is done such that the residuals of equation (95) remain unchanged. Therefore, the corrections in the second step are governed by the following discrete equations

$$\nabla_h \cdot \delta \vec{q} = -(\nabla_h \cdot \vec{q} - s) \quad (96)$$

$$-\Delta_h \delta \vec{q} + \nabla_h \delta p = 0 \quad (97)$$

where  $\nabla_h$  and  $\Delta_h$  are the discrete gradient and Laplacian operators on the given grid.

To solve equations (96) and (97), one can introduce a discrete potential function.

$$\delta \vec{q} = \nabla_h \phi \quad (98)$$

Therefore,

$$\Delta_h \phi = \delta p = -(\nabla_h \cdot \vec{q} - s) \quad (99)$$

The discrete equation (99) is used to update the pressure and the velocity fields.

In the first step of such calculations, the vorticity field is established. The potential correction in the second step does not alter the vorticity field, but it enforces the conservation of mass.

These ideas are as old as the pressure correction methods for the solution of Navier Stokes equations of incompressible flows, introduced by Harlow and Welsh (48), Chorin (49), Temam (50), Patankar and Spalding (51), and others. Kim and Moin (52) used a similar decomposition for the continuous problem. Obviously, the discrete potential is preferred to avoid the difficulties associated with the boundary conditions for the intermediate variables.

Returning to the transonic flow problem, one may consider the nonlinear Cauchy-Riemann problem

$$(\rho u)_x + (\rho u)_y = 0 \quad (100)$$

$$u_y - v_x = -\omega \quad (101)$$

where  $\rho(u, v)$ .

The correction to the velocity field can be split into two parts, irrotational and rotational. These components can be calculated in terms of potential and stream functions. An important issue in this discussion is the treatment of shocks (and wakes). Fitting procedures are needed to calculate the entropy jump and the entropy gradient behind curved shocks.

On the other hand, if the momentum equations are used to update the velocity field, and conservation of mass is enforced by a potential correction, it is not clear how to correct the pressure. Recently, there are some attempts to use the pressure correction methods for compressible flow calculations. The relation between the pressure correction and the velocity correction is, however, artificial. Still convergent results may be obtained for steady problems.

### VISCOUS FLOW SIMULATION

In the previous sections, the velocity/vorticity formulation and the pressure correction methods for the solution of the Navier Stokes equations of incompressible viscous flows are discussed. Direct applications of vector potential methods are examined next. For example, in ref. (53), the following decomposition is used:

$$u = \phi_x + \tilde{\psi}_y/\rho, \quad v = \phi_y - \tilde{\psi}_x/\rho \quad (102)$$

The continuity equation gives

$$(\rho\phi_x)_x + (\rho\phi_y)_y = 0 \quad (103)$$

with the boundary condition

$$\phi_n = \tilde{\psi}_s/\rho \quad (104)$$

where  $n$  and  $s$  are the normal and the tangential direction to the surface. The equation for  $\tilde{\psi}$  is

$$(\tilde{\psi}_{x/\rho})_x + (\tilde{\psi}_{y/\rho})_y = -\omega \quad (105)$$

with the boundary condition

$$\tilde{\psi}_{n/\rho} = -\phi_s \quad (106)$$

where  $\omega$  is obtained from the vorticity transport equation. It is assumed that the vorticity outside the viscous layer is negligible and hence a potential model is adequate.

The boundary conditions (104) and (106) represent the coupling between a viscous and an inviscid problem. The perturbative stream function behaves like a displacement thickness which derives the potential calculations. On the other hand, the pressure distribution ( $\phi_x$ ) derives the viscous calculations through the no slip boundary condition. No boundary-layer approximations have been made and

the formulation can be completely equivalent to the Navier Stokes equations. Only an inviscid grid is needed for the potential calculations, while the  $\tilde{\psi}$ - $\omega$  system is restricted to the viscous layer. Therefore, this formulation is a special form of domain decomposition techniques.

Further approximations lead to some simplifications. For example, the density in the potential equation (103) can be calculated in terms of formula (1') and therefore the pressure, according to the isentropic relation is

$$p = \rho_i^{\gamma/\gamma M_\infty^2} \quad (107)$$

On the other hand, the density in the  $\tilde{\psi}$ - $\omega$  system can be evaluated from the total enthalpy relation

$$H = \frac{\gamma p}{(\gamma-1)\rho_v} + \frac{1}{2} (u^2 + v^2) \quad (108)$$

In the transonic regime, H can be assumed constant even in the viscous layer, otherwise H can be evaluated from the energy equation. Unlike  $\rho_i$ , the density  $\rho_v$  has a boundary-layer-type profile like the velocity and the temperature. The pressure, however, can be assumed to be the same in the inviscid and viscous calculations.

One way to implement this formulation is to use, in the viscous layer, the full  $\psi$ - $\omega$  system, with the surface boundary conditions,  $\psi=0$  and  $\psi_n=0$ . The outer boundary conditions are  $\omega=0$  and  $\psi_n = \rho \phi_s$ . Equation (108) is used for the density where the pressure is assumed known.

The next step is to construct  $\tilde{\psi}$ . Equation (105) is solved with the surface boundary condition given by (106). The boundary condition at the edge of the viscous layer is  $\tilde{\psi} = 0$  and  $\omega$  is known from the first step. The output of this calculation is  $\tilde{\psi}$  on the surface.

Finally, the potential equation is solved and the process is repeated until convergence. The problem for  $\tilde{\psi}$  in the second step may be solved coupled with the potential equation to avoid convergence difficulties. In this case, the  $\psi$ - $\omega$  system provides  $\omega$  to the  $\tilde{\psi}$ - $\phi$  system, while the latter feeds back the pressure field and  $\phi_s$  at the edge of the viscous layer.

Extension to three-dimensional flows are possible using multiple stream functions. Alternatively, the viscous calculations can be based on the velocity/vorticity formulation. The role of the potential is simply to provide an approximation for the pressure field, which is needed to calculate the density in compressible flow cases. For incompressible flows, the potential can be dropped, since the pressure is eliminated from both the stream function and the velocity/vorticity formulations. The only advantage of keeping the potential is to restrict the domain of the viscous calculations.

Recently, El Dabaghi (54) obtained finite-element solutions of Navier Stokes equations using a rotational correction to the potential field. He used the decomposition

$$\vec{q} = \nabla \phi + \tilde{q}, \text{ where } \nabla \cdot \tilde{q} = 0 \quad (109)$$



He solved the following potential equation

$$\nabla \cdot \rho \nabla \phi = - \nabla \rho \cdot \vec{q} \quad (110)$$

with the surface boundary condition  $\phi_n = 0$ . The equations for  $\vec{q}$  are very similar to incompressible Navier Stokes equations. In this formulation, the grid for  $\phi$  problem must be fine enough to allow for the resolution of the nonhomogeneous term  $\nabla \rho \cdot \vec{q}$ . Similar ideas were introduced earlier by Dodge (55), Briley (56), Dwoyer (57) and others.

Also, Ecer et al.(58) extended their formulation to treat viscous flows. Their approach is applicable for inviscid flow in the limit of high Reynolds numbers. Also, outside the viscous layer, the potential formulation is recovered.

A composite velocity procedure for potential, Euler and Navier Stokes equations was introduced by Rubin et al. (59). The following velocity decomposition was proposed.

$$u = (U + 1)\phi_x \quad (111)$$

$$v = \phi_y \quad (112)$$

The multiplicative composite form of the axial velocity component consists of the potential component modified by the viscous velocity. The continuity and the tangential momentum equation are solved in a coupled manner to update  $\phi$  and  $U$ . The normal momentum equation in nonconservative form, is used to calculate the entropy. Numerical results for nonlifting airfoils indicate the differences between potential, Euler and Navier Stokes solutions.

Helmholtz decomposition, vorticity generation and trailing-edge condition for incompressible inviscid and viscous flows are discussed by Morino(60), (61). He derived also boundary integral equations for unsteady viscous and inviscid flows. In this work, the solution of the Poisson's equations,  $\nabla^2 \vec{\Psi} = -\vec{\omega}$  is given by

$$\vec{\Psi} = \frac{1}{4\pi} \int_V \frac{\vec{\omega}}{r} dV \quad (113)$$

The rotational component of the velocity vector is then

$$\vec{q}_v = \frac{-1}{4\pi} \int_V \frac{\vec{r} \times \vec{\omega}}{r^3} dV \quad (114)$$

The integral equation approach is also adopted by Kandil and Yates (62) to calculate inviscid vortical transonic flows, where both shocks and wakes are fitted.

In this regard, the work of Wu (63), on numerical boundary conditions for viscous flow problems should be mentioned. Applications to multiple-body problems are given in ref. (64).

## CONCLUSIONS

Unfortunately, no satisfactory results, based on vector potential methods, are available, so far, for real aerodynamic problems. For example, the simulation of transonic flow over a wing, with proper treatment of shocks and wakes, is still missing. At the same time, efforts on conservation laws gain momentum from recent mathematical developments in this field. Nevertheless, one can say, there is a great potential in the alternate formulations, particularly if practical methods for shock (and wake) fitting\* are developed.

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